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Asymptotic Behavior of Mayer Cluster Sums for the One-Dimensional Ising Model

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The properties of the high-field polynomials $L_n(u)$ for the one-dimensional spin $\frac{1}{2}$ Ising model are investigated. [The polynomials $L_n(u)$ are essentially lattice gas analogues of the Mayer cluster integrals $b_n(T)$ for a continuum gas.] It is shown that $u^{-1}L_n(u)$ can be expressed in terms of a shifted Jacobi polynomial of degree $n-1$. From this result it follows that $\{u^{-1}L_n(u); n=1, 2,...\}$ is a set of orthogonal polynomials in the interval $(0, 1)$ with a weight function $\omega(u)=u$, and $u^{-1}L_n(u)$ has $n-1$ simple zeros $\{u_n(v);\ v=1, 2,..., n-1\}$ which all lie in the interval $0<u<1$. Next the detailed behavior of $L_n(u)$ as $n\to\infty$ is studied. In particular, various asymptotic expansions for $L_n(u)$ are derived which are *uniformly* valid in the intervals $u<0$, $0<u<1$, and $u>1$. These expansions are then used to analyze the asymptotic properties of the zeros $\{u_n(v);\ v=1, 2,..., n-1\}$. It is found that

$$u_n(v)\sim \tfrac{1}{4}(j_{1,v}/n)^2\,[1-(j_{1,v}^2/12)\,n^{-2}+(j_{1,v}^2/720)(-3+2j_{1,v}^2)\,n^{-4}$$
$$+(j_{1,v}^2/20160)(40+4j_{1,v}^2-j_{1,v}^4)\,n^{-6}+\cdots]$$
$$u_n(n-v)\sim 1-(j_{0,v}^2/4)\,n^{-2}+(j_{0,v}^2/48)(-2+j_{0,v}^2)\,n^{-4}$$
$$+(j_{0,v}^2/2880)(2+9j_{0,v}^2-2j_{0,v}^4)\,n^{-6}+\cdots$$

as $n\to\infty$ with v *fixed*, where $j_{k,v}$ denotes the vth zero of the Bessel function $J_k(z)$.

KEY WORDS: One-dimensional Ising model; high-field polynomials; Mayer cluster sums; asymptotic analysis; zeros.

1. INTRODUCTION

The spin $\frac{1}{2}$ Ising model of a ferromagnet on a d-dimensional lattice Ω_d with N-sites has the Hamiltonian

$$\mathscr{H} = -J\sum_{(ij)} \sigma_i\sigma_j - m_0 B \sum_{i=1}^{N} \sigma_i \tag{1.1}$$

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where the first summation is taken over all nearest neighbor pairs (ij) in the lattice Ω_d, B is the magnetic field, $\sigma_i = \pm 1$, and J, m_0 are positive constants. (A comprehensive review of the Ising model has been given by Domb.[1]) In the thermodynamic limit $N \to \infty$ the free energy per spin $g(T, B)$ for the system is given by

$$-g(T, B)/k_B T = \lim_{N \to \infty} \frac{1}{N} \ln Z_N(T, B) \tag{1.2}$$

where

$$Z_N(T, B) = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \exp(-\mathscr{H}/k_B T) \tag{1.3}$$

is the partition function.

A standard procedure for investigating the thermodynamic properties of the Ising model is to expand $g(T, B)$ as the high-field series[2]

$$-g(T, B)/k_B T = -\frac{q}{8} \ln u - \frac{1}{2} \ln \mu + \sum_{n=1}^{\infty} L_n(u)\, \mu^n \tag{1.4}$$

where

$$u = \exp(-4J/k_B T) \tag{1.5}$$

$$\mu = \exp(-2m_0 B/k_B T) \tag{1.6}$$

and q is the coordination number of the lattice Ω_d. The coefficient $L_n(u)$ is a polynomial of degree $nq/2$ in the variable u except when n and q are both odd. [For this special case $L_n(u)$ is a polynomial of degree nq in the variable $u^{1/2}$.] It can also be shown that $L_n(u)$ can always be written in the form

$$L_n(u) = u^{qn/2} Q_n(1/u) \tag{1.7}$$

where $Q_n(1/u)$ is a polynomial in the variable $1/u$. If we interpret the Ising model as a model of a lattice gas,[3] we find that the polynomial $Q_n(1/u)$ is a lattice analogue of the Mayer cluster integral $b_n(T)$ which occurs in the activity expansion for the pressure of an imperfect gas. Sykes *et al.*[4–8] have used sophisticated graph-theoretic methods to derive explicit expressions for a considerable number of the initial coefficients $L_n(u)$ for a variety of two- and three-dimensional lattices.

The expansion (1.4) is of particular interest in the theory of phase transitions because the asymptotic behavior of $L_n(u)$ as $n \to \infty$, with u fixed and T less than the critical temperature T_c, essentially determines the behavior of the free energy $g(T, B)$ in the neighborhood of the phase boundary $\mu = 1$. Although no exact asymptotic analysis of $L_n(u)$ has yet been carried out, a number of interesting approximate theories have been developed. Essam and Fisher[9] and Fisher[10,11] have used heuristic arguments based on the droplet model of condensation to derive the following asymptotic representation for $L_n(u)$:

$$L_n(u) \sim A n^{-s\tau} (\lambda u^{1/2})^{Dn^s} \tag{1.8}$$

as $n \to \infty$, where A, D, λ, τ, and s are constants with $0 < s < 1$. This asymptotic formula is inconsistent with the classical theory of condensation and the numerical work of Gaunt and Baker[12] on the metastable state of the Ising model because it gives rise to an essential singularity in the free energy on the phase boundary $B = 0$, $T < T_c$.[10,11,13] Following the work of Fisher, there have been several empirical attempts to modify the droplet model formulation and to extend its range of validity.[14-16] Attempts have also been made by Domb and Guttmann[17] and Domb[18,19] to reconcile the classical theory of condensation with the conflicting predictions of the droplet model by developing a more systematic diagrammatic approach to the problem.

The main aim in this paper is to determine the detailed asymptotic behavior of $L_n(u)$ as $n \to \infty$ for the one-dimensional spin-$\frac{1}{2}$ Ising model. In particular, various uniform asymptotic expansions for $L_n(u)$ are derived by using the general methods developed by Darboux[20] and Olver.[21,22] These expansions are then used to investigate the asymptotic properties of the zeros of $L_n(u)$ as $n \to \infty$. It is hoped that the *exact* results obtained will provide *some* insight into the asymptotic behavior of $L_n(u)$ for the two- and three-dimensional Ising models.

2. PROPERTIES OF $L_n(u)$ IN ONE DIMENSION

It is readily seen from (1.4) that, in general, the magnetization per spin of the Ising model

$$m = -(\partial g/\partial B)_T \tag{2.1}$$

has a high-field series representation

$$m/m_0 = 1 - 2 \sum_{n=1}^{\infty} n L_n(u)\, \mu^n \tag{2.2}$$

For the particular case of the one-dimensional Ising model we also have the closed-form expression[2]

$$m/m_0 = (1 - \mu)[1 - 2(1 - 2u)\, \mu + \mu^2]^{-1/2} \tag{2.3}$$

If the standard generating function

$$(1-2x\mu+\mu^2)^{-1/2}=\sum_{n=0}^{\infty} P_n(x)\,\mu^n \tag{2.4}$$

is applied to (2.3) and the resulting expansion is compared with (2.2), we obtain the formula

$$L_n(u)=\frac{1}{2n}\,[P_{n-1}(1-2u)-P_n(1-2u)] \qquad (n\geqslant 1) \tag{2.5}$$

where $P_n(x)$ is the Legendre polynomial of degree n. This result has also been given by Bessis *et al.*[23]

A simplification of (2.5) can be achieved by using the relation[24]

$$[n+\tfrac{1}{2}(\alpha+\beta)](1-x)\,P_{n-1}^{(\alpha+1,\beta)}(x)=(n+\alpha)\,P_{n-1}^{(\alpha,\beta)}(x)-nP_n^{(\alpha,\beta)}(x) \tag{2.6}$$

with $\alpha=\beta=0$, and the identity

$$P_n^{(0,0)}(x)\equiv P_n(x) \tag{2.7}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of degree n. In this manner we find that

$$L_n(u)=\frac{u}{n}\,P_{n-1}^{(1,0)}(1-2u) \tag{2.8}$$

We can write (2.8) in the alternative form

$$L_n(u)=\frac{u}{n}\,(-1)^n\,R_{n-1}^{(0,1)}(u) \tag{2.9}$$

where $R_n^{(\alpha,\beta)}(u)$ denotes the *shifted* Jacobi polynomial of degree n.[25] In the notation of Magnus and Oberhettinger,[26] we also have

$$L_n(u)=u\mathscr{F}_{n-1}(2,2,u) \tag{2.10}$$

where

$$\mathscr{F}_n(\alpha,\gamma,u)={}_2F_1(-n,n+\alpha;\gamma;u) \tag{2.11}$$

and ${}_2F_1$ is the hypergeometric function.

We can now use the standard theory of Jacobi polynomials[24,25,27] to obtain the Rodrigues' formula

$$L_n(u)=\frac{1}{n!}\,D^{n-1}[u^n(1-u)^{n-1}] \tag{2.12}$$

and the three-term recurrence relation

$$(2n-1)(n+1)^2 L_{n+1}(u) - 2n[2n^2 - (4n^2-1)u] L_n(u) + (2n+1)(n-1)^2 L_{n-1}(u) = 0 \tag{2.13}$$

where $D \equiv d/du$ and $n = 1, 2, 3, \ldots$. From the orthogonality property

$$\int_0^1 R_n^{(0,1)}(u) R_m^{(0,1)}(u)\, u\, du = \tfrac{1}{2}(n+1)^{-1} \delta_{mn} \tag{2.14}$$

we see that $\{u^{-1}L_n(u); n = 1, 2, 3, \ldots\}$ is a set of orthogonal polynomials in the interval (0, 1) with a weight function $\omega(u) = u$. It is also evident that all the $(n-1)$ zeros of the polynomial $u^{-1}L_n(u)$ are simple, and located in the interval $0 < u < 1$. More generally, the numerical work of Gaunt[28] indicates that the coefficient $L_n(u)$ for the two- and three-dimensional spin $\frac{1}{2}$ Ising models always has $(n-1)$ simple zeros in the interval $u_c < u < 1$, where u_c is the critical value of u. (Note that for the Ising model in one dimension $u_c \equiv 0$.)

3. DARBOUX ANALYSIS OF $L_n(u)$ IN ONE DIMENSION

In this section the method of Darboux[20] will be used to determine the asymptotic behavior of $L_n(u)$ as $n \to \infty$ with u *fixed* and negative. Similar results will also be given for the case $u > 1$. We begin by writing the generating function (2.4) with $x = 1 - 2u$ in the alternative form

$$[(\mu - e^{\vartheta_0})(\mu - e^{-\vartheta_0})]^{-1/2} = \sum_{n=0}^{\infty} P_n(\cosh \vartheta_0)\, \mu^n \tag{3.1}$$

where the parameter $\vartheta_0 = \vartheta_0(u)$ is defined by

$$\vartheta_0 = \operatorname{arc\,cosh}(1-2u) \tag{3.2}$$

with $u < 0$ and $\vartheta_0 > 0$. Next we carry out the change of variable

$$\mu = e^{-\vartheta_0}(1-h) \tag{3.3}$$

and expand the left-hand side of (3.1) in powers of h about the dominant branch-point singularity at $\mu = e^{-\vartheta_0}$. This procedure yields

$$(\tfrac{1}{2}e^{\vartheta_0} \operatorname{csch} \vartheta_0)^{1/2} \sum_{m=0}^{\infty} \binom{-1/2}{m} (\tfrac{1}{2}e^{-\vartheta_0} \operatorname{csch} \vartheta_0)^m (1 - \mu e^{\vartheta_0})^{m-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(\cosh \vartheta_0)\, \mu^n \tag{3.4}$$

where $\binom{-1/2}{m}$ is a binomial coefficient. If we now formally expand the left-hand side of (3.4) in powers of μ and equate the coefficients of μ^n we obtain the asymptotic formula

$$P_n(\cosh \vartheta_0) \sim (-1)^n e^{n\vartheta_0}(\tfrac{1}{2}e^{\vartheta_0} \operatorname{csch} \vartheta_0)^{1/2} \times \sum_{m=0}^{\infty} \binom{-1/2}{m}\binom{m-\frac{1}{2}}{n} (\tfrac{1}{2}e^{-\vartheta_0} \operatorname{csch} \vartheta_0)^m \tag{3.5}$$

as $n \to \infty$, with fixed $u < 0$.

It is possible to express the expansion (3.5) in terms of the ${}_2F_1$ hypergeometric series by writing the two binomial coefficients in terms of gamma functions. The final result is

$$P_n(\cosh \vartheta_0) \sim e^{n\vartheta_0}(\tfrac{1}{2}e^{\vartheta_0} \operatorname{csch} \vartheta_0)^{1/2} [(\tfrac{1}{2})_n/n!] \times {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}-n; -\tfrac{1}{2}e^{-\vartheta_0} \operatorname{csch} \vartheta_0) \tag{3.6}$$

as $n \to \infty$, with fixed $u < 0$, where

$$(\tfrac{1}{2})_n = \Gamma(n+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}) \tag{3.7}$$

From the formula (3.6) we can readily derive the basic asymptotic expansion

$$P_n(\cosh \vartheta_0) \sim (2\pi n \sinh \vartheta_0)^{-1/2} e^{(n+\frac{1}{2})\vartheta_0} \sum_{m=0}^{\infty} a_m(\vartheta_0)(8n)^{-m} \tag{3.8}$$

as $n \to \infty$, with fixed $u < 0$, where

$$a_0(\vartheta_0) = 1 \tag{3.9}$$

$$a_1(\vartheta_0) = -(2 - \coth \vartheta_0) \tag{3.10}$$

$$a_2(\vartheta_0) = \tfrac{1}{2}(4 - 12 \coth \vartheta_0 + 9 \coth^2 \vartheta_0) \tag{3.11}$$

$$a_3(\vartheta_0) = \tfrac{5}{2}(8 - 4 \coth \vartheta_0 - 18 \coth^2 \vartheta_0 + 15 \coth^3 \vartheta_0) \tag{3.12}$$

$$a_4(\vartheta_0) = -\tfrac{21}{8}(16 - 160 \coth \vartheta_0 + 120 \coth^2 \vartheta_0 + 200 \coth^3 \vartheta_0 - 175 \coth^4 \vartheta_0) \tag{3.13}$$

If we make the replacement $n \to n-1$ in (3.8), we obtain the similar expansion

$$P_{n-1}(\cosh \vartheta_0) \sim (2\pi n \sinh \vartheta_0)^{-1/2} e^{(n-\frac{1}{2})\vartheta_0} \sum_{m=0}^{\infty} d_m(\vartheta_0)(8n)^{-m} \tag{3.14}$$

as $n \to \infty$, with fixed $u < 0$, where

$$d_0(\vartheta_0) = 1 \tag{3.15}$$

$$d_1(\vartheta_0) = (2 + \coth \vartheta_0) \tag{3.16}$$

$$d_2(\vartheta_0) = \tfrac{1}{2}(4 + 12 \coth \vartheta_0 + 9 \coth^2 \vartheta_0) \tag{3.17}$$

$$d_3(\vartheta_0) = -\tfrac{5}{2}(8 + 4 \coth \vartheta_0 - 18 \coth^2 \vartheta_0 - 15 \coth^3 \vartheta_0) \tag{3.18}$$

$$d_4(\vartheta_0) = -\tfrac{21}{8}(16 + 160 \coth \vartheta_0 + 120 \coth^2 \vartheta_0 - 200 \coth^3 \vartheta_0 - 175 \coth^4 \vartheta_0) \tag{3.19}$$

We now substitute the expansions (3.8) and (3.14) in the formula (2.5). This procedure yields the important result

$$L_n(u) \sim -\tfrac{1}{2}\pi^{-1/2} n^{-3/2} [\tanh(\vartheta_0/2)]^{1/2} e^{n\vartheta_0} \sum_{m=0}^{\infty} f_m(\vartheta_0)(8n \sinh \vartheta_0)^{-m} \tag{3.20}$$

as $n \to \infty$, with fixed $u < 0$, where

$$f_0(\vartheta_0) = 1 \tag{3.21}$$

$$f_1(\vartheta_0) = -(2 + \cosh \vartheta_0) \tag{3.22}$$

$$f_2(\vartheta_0) = -\tfrac{1}{2}(4 + 12 \cosh \vartheta_0 - \cosh^2 \vartheta_0) \tag{3.23}$$

$$f_3(\vartheta_0) = -\tfrac{5}{2}(8 + 4 \cosh \vartheta_0 + 10 \cosh^2 \vartheta_0 - \cosh^3 \vartheta_0) \tag{3.24}$$

$$f_4(\vartheta_0) = -\tfrac{21}{8}(16 + 160 \cosh \vartheta_0 + 8 \cosh^2 \vartheta_0 + 40 \cosh^3 \vartheta_0 + \cosh^4 \vartheta_0) \tag{3.25}$$

Finally, we investigate the behavior of $L_n(u)$ as $n \to \infty$ with fixed $u > 1$. For this case we write (2.5) in the alternative form

$$L_n(u) = \frac{(-1)^{n-1}}{2n} [P_{n-1}(\cosh \vartheta_1) + P_n(\cosh \vartheta_1)] \tag{3.26}$$

where

$$\vartheta_1 = \operatorname{arc} \cosh(2u - 1) \tag{3.27}$$

with $u > 1$ and $\vartheta_1 > 0$, and then apply the expansions (3.8) and (3.14). In this manner we find that

$$L_n(u) \sim \tfrac{1}{2}(-1)^{n-1} \pi^{-1/2} n^{-3/2} [\coth(\vartheta_1/2)]^{1/2} e^{n\vartheta_1} \times \sum_{m=0}^{\infty} g_m(\vartheta_1)(8n \sinh \vartheta_1)^{-m} \tag{3.28}$$

as $n \to \infty$, with fixed $u > 1$, where

$$g_0(\vartheta_1) = 1 \tag{3.29}$$

$$g_1(\vartheta_1) = (2 - \cosh \vartheta_1) \tag{3.30}$$

$$g_2(\vartheta_1) = -\tfrac{1}{2}(4 - 12 \cosh \vartheta_1 - \cosh^2 \vartheta_1) \tag{3.31}$$

$$g_3(\vartheta_1) = \tfrac{5}{2}(8 - 4 \cosh \vartheta_1 + 10 \cosh^2 \vartheta_1 + \cosh^3 \vartheta_1) \tag{3.32}$$

$$g_4(\vartheta_1) = -\tfrac{21}{8}(16 - 160 \cosh \vartheta_1 + 8 \cosh^2 \vartheta_1 - 40 \cosh^3 \vartheta_1 + \cosh^4 \vartheta_1) \tag{3.33}$$

The basic asymptotic expansions (3.20) and (3.28) are only applicable in the *nonphysical* intervals $u < 0$ and $u > 1$, respectively. Furthermore, these expansions clearly break down when n is large and $n\vartheta_i$ is *small*, where $i = 0, 1$. However, we shall find in the following sections that they play a crucial role in the derivation of uniform asymptotic expansions for $L_n(u)$ which have a wider range of validity.

4. UNIFORM ASYMPTOTIC EXPANSIONS FOR $L_n(u)$

In this section we shall use the methods of Olver[21,22] to derive an asymptotic expansion for $L_n(u)$ which is uniform with respect to the variable u when u lies in the interval $u < 0$. A similar result which is valid in the interval $u > 1$ will also be given.

We begin by considering the standard differential equation[24]

$$(1 - x^2)\, D^2 P_n^{(1,0)}(x) - (1 + 3x)\, D P_n^{(1,0)}(x) + n(n+2)\, P_n^{(1,0)}(x) = 0 \tag{4.1}$$

where $D \equiv d/dx$. If we reduce (4.1) to its normal form,[29] we find that

$$y(x) = (1 - x)(1 + x)^{1/2}\, P_n^{(1,0)}(x) \tag{4.2}$$

satisfies the simplified differential equation

$$D^2 y + [(n+1)^2\, (1 - x^2)^{-1} + \tfrac{1}{4}(1 + x)^{-2}]\, y = 0 \tag{4.3}$$

From this result and the formula (2.8) we readily see that

$$w(u) = (1 - u)^{1/2}\, L_n(u) \tag{4.4}$$

is a solution of the differential equation

$$\frac{d^2 w}{du^2} + [n^2 u^{-1}(1 - u)^{-1} + \tfrac{1}{4}(1 - u)^{-2}]\, w = 0 \tag{4.5}$$

It is interesting to note that (4.5) has *transition* points[21] at $u = 0$ and 1.

Next we change the variables (u, w) in (4.5) by using the transformations

$$u = \tfrac{1}{2}(1 - \cosh \vartheta_0) \tag{4.6}$$

$$w = [(\sinh \vartheta_0)/\vartheta_0]^{1/2}\, W \tag{4.7}$$

This procedure yields the further differential equation

$$\frac{d^2 W}{d\vartheta_0^2} = \frac{1}{\vartheta_0}\frac{dW}{d\vartheta_0} + [n^2 + \bar{f}(\vartheta_0)]\, W \tag{4.8}$$

where

$$\bar{f}(\vartheta_0) = \tfrac{1}{4}[-(3/\vartheta_0^2) + \operatorname{csch}^2 \vartheta_0\, (1 + 2\cosh \vartheta_0)] \tag{4.9}$$

and $\vartheta_0 > 0$. The function $\bar{f}(\vartheta_0)$ is *analytic* at $\vartheta_0 = 0$ provided that we define $\bar{f}(0) \equiv 0$. We now apply to (4.8) a theorem proved by Olver.[22] In this manner we obtain the uniform asymptotic expansion

$$L_n(u) \sim C_n[(\vartheta_0/2)\tanh(\vartheta_0/2)]^{1/2}\Bigg[I_1(n\vartheta_0)\sum_{s=0}^{\infty} D_{2s}^{(0)}(\vartheta_0)(8n)^{-2s} + I_2(n\vartheta_0)\sum_{s=0}^{\infty} D_{2s+1}^{(0)}(\vartheta_0)(8n)^{-2s-1}\Bigg] \tag{4.10}$$

as $n \to \infty$ with $\vartheta_0 > 0$ and $u < 0$, where $I_\nu(z)$ denotes a modified Bessel function of order ν, C_n only depends on the integer n, and the coefficient $D_0^{(0)}(\vartheta_0) \equiv 1$. The result (4.10) is much more powerful than the Darboux expansion (3.20) and gives an accurate approximation for $L_n(u)$ when n is large and $n\vartheta_0$ has *any* value in the interval $(0, \infty)$.

Olver[22] has shown that the coefficients $D_m^{(0)}(\vartheta_0)$ $(m = 1, 2,...)$ in (4.10) can be generated, at least in principle, by using two coupled integral recurrence relations with the initial condition $D_0^{(0)}(\vartheta_0) \equiv 1$. For the simplest case, we find

$$D_1^{(0)}(\vartheta_0) = 4\int_0^{\vartheta_0} \bar{f}(t)\, dt \tag{4.11}$$

where the function $\bar{f}$ is defined in (4.9). The evaluation of this integral gives the formula

$$D_1^{(0)}(\vartheta_0) = \vartheta_0^{-1}[3 - (\vartheta_0/\sinh \vartheta_0)(2 + \cosh \vartheta_0)] \tag{4.12}$$

It is not feasible to carry out this procedure for the higher-order coefficients because of the large amount of algebra which is involved.

Fortunately, it is also possible to determine the coefficients $D_m^{(0)}(\vartheta_0)$ ($m=1, 2,\ldots$) by replacing the Bessel functions in (4.10) with the asymptotic representation[27]

$$I_\nu(z) \sim (2\pi z)^{-1/2} e^z \sum_{m=0}^{\infty} (-1)^m (\nu, m)(2z)^{-m} \tag{4.13}$$

as $z \to \infty$, where

$$(\nu, m) = (2^{2m} m!)^{-1} \prod_{k=1}^{m} [4\nu^2 - (2k-1)^2] \qquad (m \geqslant 1) \tag{4.14}$$

with $(\nu, 0) \equiv 1$. A comparison of the resulting expansion with the equivalent Darboux expansion (3.20) enables one to obtain the required formulas for $D_m^{(0)}(\vartheta_0)$ ($m=1, 2,\ldots$). The final results are

$$D_0^{(0)}(\vartheta_0) = 1 \tag{4.15}$$

$$D_1^{(0)}(\vartheta_0) = \vartheta_0^{-1}[3 + \Delta_0 f_1(\vartheta_0)] \tag{4.16}$$

$$D_2^{(0)}(\vartheta_0) = (2\vartheta_0^2)^{-1} [105 + 30\Delta_0 f_1(\vartheta_0) + 2\Delta_0^2 f_2(\vartheta_0)] \tag{4.17}$$

$$D_3^{(0)}(\vartheta_0) = (2\vartheta_0^3)^{-1} [105 - 15\Delta_0 f_1(\vartheta_0) + 6\Delta_0^2 f_2(\vartheta_0) + 2\Delta_0^3 f_3(\vartheta_0)] \tag{4.18}$$

$$D_4^{(0)}(\vartheta_0) = (8\vartheta_0^4)^{-1} [10395 - 1260\Delta_0 f_1(\vartheta_0) + 420\Delta_0^2 f_2(\vartheta_0) + 120\Delta_0^3 f_3(\vartheta_0) + 8\Delta_0^4 f_4(\vartheta_0)] \tag{4.19}$$

where

$$\Delta_0 \equiv \vartheta_0 / \sinh \vartheta_0 \tag{4.20}$$

This analysis also yields the additional result

$$C_n = -1/n \tag{4.21}$$

A uniform asymptotic expansion for $L_n(u)$ which is valid for $u > 1$ can be derived by first applying the transformations

$$u = \tfrac{1}{2}(1 + \cosh \vartheta_1) \tag{4.22}$$

$$w = [(\sinh \vartheta_1)/\vartheta_1]^{1/2} W \tag{4.23}$$

to (4.5). This procedure leads to the differential equation

$$\frac{d^2W}{d\vartheta_1^2}=\frac{1}{\vartheta_1}\frac{dW}{d\vartheta_1}+\left[n^2-\frac{1}{\vartheta_1^2}+\bar{g}(\vartheta_1)\right]W \tag{4.24}$$

where

$$\bar{g}(\vartheta_1)=\tfrac{1}{4}[\vartheta_1^{-2}+\operatorname{csch}^2\vartheta_1\,(1-2\cosh\vartheta_1)] \tag{4.25}$$

and $\vartheta_1>0$. The function $\bar{g}(\vartheta_1)$ is *analytic* at $\vartheta_1=0$ provided that we define $\bar{g}(0)\equiv -1/6$. We now apply to (4.24) the theorem D proved by Olver.[22] This procedure yields the uniform asymptotic expansion

$$L_n(u)\sim\bar{C}_n[(\vartheta_1/2)\coth(\vartheta_1/2)]^{1/2}\left[I_0(n\vartheta_1)\sum_{s=0}^{\infty}D_{2s}^{(1)}(\vartheta_1)(8n)^{-2s}\right.$$
$$\left.+I_1(n\vartheta_1)\sum_{s=0}^{\infty}D_{2s+1}^{(1)}(\vartheta_1)(8n)^{-2s-1}\right] \tag{4.26}$$

as $n\to\infty$ with $\vartheta_1>0$ and $u>1$, where $\bar{C}_n$ only depends on the integer n and the coefficient $D_0^{(1)}(\vartheta_1)\equiv 1$.

The coefficients $D_m^{(1)}(\vartheta_1)$ $(m=1,2,...)$ and $\bar{C}_n$ may be determined by substituting (4.13) in (4.26). If the resulting expansion is compared with the Darboux expansion (3.28), we find that

$$\bar{C}_n=(-1)^{n-1}/n \tag{4.27}$$

$$D_0^{(1)}(\vartheta_1)=1 \tag{4.28}$$

$$D_1^{(1)}(\vartheta_1)=\vartheta_1^{-1}[-1+\varDelta_1 g_1(\vartheta_1)] \tag{4.29}$$

$$D_2^{(1)}(\vartheta_1)=(2\vartheta_1^2)^{-1}[-15+6\varDelta_1 g_1(\vartheta_1)+2\varDelta_1^2 g_2(\vartheta_1)] \tag{4.30}$$

$$D_3^{(1)}(\vartheta_1)=(2\vartheta_1^3)^{-1}[-75+9\varDelta_1 g_1(\vartheta_1)-2\varDelta_1^2 g_2(\vartheta_1)$$
$$+2\varDelta_1^3 g_3(\vartheta_1)] \tag{4.31}$$

$$D_4^{(1)}(\vartheta_1)=(8\vartheta_1^4)^{-1}[-4725+420\varDelta_1 g_1(\vartheta_1)-60\varDelta_1^2 g_2(\vartheta_1)$$
$$+24\varDelta_1^3 g_3(\vartheta_1)+8\varDelta_1^4 g_4(\vartheta_1)] \tag{4.32}$$

where

$$\varDelta_1\equiv\vartheta_1/\sinh\vartheta_1 \tag{4.33}$$

It should be noted that the coefficients $D_m^{(k)}(\vartheta_k)$ $(k=0,1;\ m=1,2,...)$ in the uniform expansions (4.10) and (4.26) are all *well-behaved* functions of ϑ_k in the limit $\vartheta_k\to 0$, (more precisely, the coefficients have *removable* singularities at $\vartheta_k=0$).

It has been shown by Olver[22] that the basic asymptotic expansion (4.10) can also be written in the alternative form

$$L_n(u) \sim -(1/n)[(\vartheta_0/2)\tanh(\vartheta_0/2)]^{1/2}\left[1+\sum_{s=1}^{\infty} A_{2s}^{(0)}(\vartheta_0)(8n)^{-2s}\right]$$

$$\times I_1\left[n\vartheta_0 + n\vartheta_0 \sum_{s=1}^{\infty} B_{2s}^{(0)}(\vartheta_0)(8n)^{-2s}\right] \tag{4.34}$$

as $n \to \infty$, with $\vartheta_0 > 0$ and $u < 0$. The coefficients $A_{2s}^{(0)}(\vartheta_0)$ and $B_{2s}^{(0)}(\vartheta_0)$ may be related to the coefficients $D_m^{(0)}(\vartheta_0)$ $(m = 1, 2,...)$ in (4.10) by expanding the Bessel function in (4.34) as a Taylor series. Standard recurrence relations are then used to express the derivatives $I_1^{(r)}(n\vartheta_0)$ in terms of $I_1(n\vartheta_0)$ and $I_2(n\vartheta_0)$. The final results are

$$B_2^{(0)}(\vartheta_0) = 8\vartheta_0^{-1} D_1^{(0)}(\vartheta_0) \tag{4.35}$$

$$A_2^{(0)}(\vartheta_0) = D_2^{(0)}(\vartheta_0) - \tfrac{1}{2}[D_1^{(0)}(\vartheta_0)]^2 - 8\vartheta_0^{-1} D_1^{(0)}(\vartheta_0) \tag{4.36}$$

$$B_4^{(0)}(\vartheta_0) = 8\vartheta_0^{-1} D_3^{(0)}(\vartheta_0) - 8\vartheta_0^{-1} D_1^{(0)}(\vartheta_0)\, D_2^{(0)}(\vartheta_0)$$
$$+ \tfrac{8}{3}\vartheta_0^{-1}[D_1^{(0)}(\vartheta_0)]^3 + 96\vartheta_0^{-2}[D_1^{(0)}(\vartheta_0)]^2 \tag{4.37}$$

The application of similar methods to (4.26) yields the further asymptotic representation

$$L_n(u) \sim (-1)^{n-1} n^{-1}[(\vartheta_1/2)\coth(\vartheta_1/2)]^{1/2}\left[1+\sum_{s=1}^{\infty} A_{2s}^{(1)}(\vartheta_1)(8n)^{-2s}\right]$$

$$\times I_0\left[n\vartheta_1 + n\vartheta_1 \sum_{s=1}^{\infty} B_{2s}^{(1)}(\vartheta_1)(8n)^{-2s}\right] \tag{4.38}$$

as $n \to \infty$, with $\vartheta_1 > 0$ and $u > 1$. The first few coefficients in this expansion are given by

$$B_2^{(1)}(\vartheta_1) = 8\vartheta_1^{-1} D_1^{(1)}(\vartheta_1) \tag{4.39}$$

$$A_2^{(1)}(\vartheta_1) = D_2^{(1)}(\vartheta_1) - \tfrac{1}{2}[D_1^{(1)}(\vartheta_1)]^2 \tag{4.40}$$

$$B_4^{(1)}(\vartheta_1) = 8\vartheta_1^{-1} D_3^{(1)}(\vartheta_1) - 8\vartheta_1^{-1} D_1^{(1)}(\vartheta_1)\, D_2^{(1)}(\vartheta_1)$$
$$+ \tfrac{8}{3}\vartheta_1^{-1}[D_1^{(1)}(\vartheta_1)]^3 + 32\vartheta_1^{-2}[D_1^{(1)}(\vartheta_1)]^2 \tag{4.41}$$

We shall find in Section 6 that the results (4.34) and (4.38) are particularly useful for analyzing the asymptotic properties of the zeros of $L_n(u)$.

5. ASYMPTOTIC BEHAVIOR OF $L_n(u)$ FOR $0<u<1$

The asymptotic properties of $L_n(u)$ in the *physically* significant interval (0, 1) can now be investigated by applying the transformation $\vartheta_0 = i\theta_0$ $(0<\theta_0<\pi)$ to the basic result (4.10). We find that

$$L_n(u) \sim n^{-1}[(\theta_0/2)\tan(\theta_0/2)]^{1/2}\left[J_1(n\theta_0)\sum_{s=0}^{\infty} E_{2s}^{(0)}(\theta_0)(8n)^{-2s} + J_2(n\theta_0)\sum_{s=0}^{\infty} E_{2s+1}^{(0)}(\theta_0)(8n)^{-2s-1}\right] \tag{5.1}$$

as $n\to\infty$, where

$$\theta_0 = \arccos(1-2u) \qquad (0<u<1) \tag{5.2}$$

$$E_{2s}^{(0)}(\theta_0) \equiv D_{2s}^{(0)}(i\theta_0) \tag{5.3}$$

$$E_{2s+1}^{(0)}(\theta_0) \equiv iD_{2s+1}^{(0)}(i\theta_0) \tag{5.4}$$

and $J_\nu(z)$ denotes a Bessel function of order ν. The first few coefficients $E_m^{(0)}(\theta_0)$ $(m=0, 1, 2,\ldots)$ are given by the explicit formulas

$$E_0^{(0)}(\theta_0) = 1 \tag{5.5}$$

$$E_1^{(0)}(\theta_0) = \theta_0^{-1}[3-\delta_0(2+\cos\theta_0)] \tag{5.6}$$

$$E_2^{(0)}(\theta_0) = -(2\theta_0^2)^{-1}[105 - 30\delta_0(2+\cos\theta_0) - \delta_0^2(4+12\cos\theta_0 - \cos^2\theta_0)] \tag{5.7}$$

$$E_3^{(0)}(\theta_0) = -(2\theta_0^3)^{-1}[105 + 15\delta_0(2+\cos\theta_0) - 3\delta_0^2(4+12\cos\theta_0-\cos^2\theta_0) - 5\delta_0^3(8+4\cos\theta_0+10\cos^2\theta_0-\cos^3\theta_0)] \tag{5.8}$$

$$E_4^{(0)}(\theta_0) = 3(8\theta_0^4)^{-1}[3465 + 420\delta_0(2+\cos\theta_0) - 70\delta_0^2(4+12\cos\theta_0-\cos^2\theta_0) - 100\delta_0^3(8+4\cos\theta_0+10\cos^2\theta_0-\cos^3\theta_0) - 7\delta_0^4(16+160\cos\theta_0+8\cos^2\theta_0 + 40\cos^3\theta_0+\cos^4\theta_0)] \tag{5.9}$$

where

$$\delta_0 \equiv \theta_0/\sin\theta_0 \tag{5.10}$$

It is evident that (5.1) provides one with a uniform expansion for $L_n(u)$ which is particularly useful in the neighborhood of the critical point $u=0$. In the limit $u \to 1-$ the expansion (5.1) breaks down because of the presence of a *transition point* at $u=1$.

This difficulty can be overcome by applying the transformation $\vartheta_1 = i\theta_1$ $(0<\theta_1<\pi)$ to the result (4.26). Hence we obtain

$$L_n(u) \sim (-1)^{n-1} n^{-1}[(\theta_1/2)\cot(\theta_1/2)]^{1/2}\left[J_0(n\theta_1)\sum_{s=0}^{\infty} E_{2s}^{(1)}(\theta_1)(8n)^{-2s} + J_1(n\theta_1)\sum_{s=0}^{\infty} E_{2s+1}^{(1)}(\theta_1)(8n)^{-2s-1}\right] \tag{5.11}$$

as $n \to \infty$, where

$$\theta_1 = \text{arc}\cos(2u-1) \qquad (0<u<1) \tag{5.12}$$

$$E_{2s}^{(1)}(\theta_1) \equiv D_{2s}^{(1)}(i\theta_1) \tag{5.13}$$

$$E_{2s+1}^{(1)}(\theta_1) \equiv iD_{2s+1}^{(1)}(i\theta_1) \tag{5.14}$$

The first few coefficients $E_m^{(1)}(\theta_1)$ $(m=0, 1, 2, \ldots)$ are

$$E_0^{(1)}(\theta_1) = 1 \tag{5.15}$$

$$E_1^{(1)}(\theta_1) = -\theta_1^{-1}[1-\delta_1(2-\cos\theta_1)] \tag{5.16}$$

$$E_2^{(1)}(\theta_1) = (2\theta_1^2)^{-1}[15-6\delta_1(2-\cos\theta_1) + \delta_1^2(4-12\cos\theta_1-\cos^2\theta_1)] \tag{5.17}$$

$$E_3^{(1)}(\theta_1) = (2\theta_1^3)^{-1}[75-9\delta_1(2-\cos\theta_1) - \delta_1^2(4-12\cos\theta_1-\cos^2\theta_1) - 5\delta_1^3(8-4\cos\theta_1+10\cos^2\theta_1+\cos^3\theta_1)] \tag{5.18}$$

$$E_4^{(1)}(\theta_1) = -3(8\theta_1^4)^{-1}[1575-140\delta_1(2-\cos\theta_1) - 10\delta_1^2(4-12\cos\theta_1-\cos^2\theta_1) - 20\delta_1^3(8-4\cos\theta_1+10\cos^2\theta_1+\cos^3\theta_1) + 7\delta_1^4(16-160\cos\theta_1+8\cos^2\theta_1 - 40\cos^3\theta_1+\cos^4\theta_1)] \tag{5.19}$$

where

$$\delta_1 \equiv \theta_1/\sin\theta_1 \tag{5.20}$$

Table I. Comparison of the Exact Values of $L_n(\frac{1}{2})$, ($n=2, 3,..., 15$) with the Corresponding Asymptotic Values[a]

n	Exact $L_n(1/2)$	$e_0(n)$	$e_1(n)$
2	1/8	7.4×10^{-5}	4.0×10^{-5}
3	−1/12	8.6×10^{-7}	1.2×10^{-5}
4	−3/64	-9.2×10^{-7}	-9.2×10^{-7}
5	3/80	-8.4×10^{-8}	-4.2×10^{-7}
6	5/192	6.5×10^{-8}	8.1×10^{-8}
7	−5/224	1.2×10^{-8}	4.7×10^{-8}
8	−35/2048	-9.8×10^{-9}	-1.4×10^{-8}
9	35/2304	-2.7×10^{-9}	-9.0×10^{-9}
10	63/5120	2.2×10^{-9}	3.4×10^{-9}
11	−63/5632	8.0×10^{-10}	2.4×10^{-9}
12	−77/8192	-6.7×10^{-10}	-1.1×10^{-9}
13	231/26624	-2.8×10^{-10}	-8.0×10^{-10}
14	429/57334	2.4×10^{-10}	4.0×10^{-10}
15	−143/20480	1.2×10^{-10}	3.1×10^{-10}

[a] The quantities $e_0(n)$ and $e_1(n)$ are the differences between the asymptotic values of $L_n(1/2)$ as determined from the formulas (5.1) and (5.11), respectively, and the exact value of $L_n(1/2)$.

The coefficients $E_m^{(k)}(\theta_k)$ ($k=0, 1; m=1, 2,...$) in the expansions (5.1) and (5.11) are all *well-behaved* functions of θ_k in the limit $\theta_k \to 0$ (more precisely, the coefficients have *removable* singularities at $\theta_k = 0$).

In order to provide a check on the analysis in this section the uniform expansions (5.1) and (5.11) have been used to calculate $L_n(u)$ for $2 \leqslant n \leqslant 15$ with $u=\frac{1}{2}$. The results are given in Table I. We see that the asymptotic approximations for $L_n(\frac{1}{2})$ are in excellent agreement with the exact value. Finally we note that *nonuniform* expansions for $L_n(u)$ which are valid in the oscillatory region $0<u<1$ can be established by substituting the standard asymptotic expansion for $J_\nu(z)$ in (5.1) and (5.11).

6. ASYMPTOTIC PROPERTIES OF ZEROS OF $L_n(u)$

The polynomial $u^{-1}L_n(u)$ has $(n-1)$ simple zeros $u_n(\nu)$ ($\nu = 1, 2,..., n-1$), which are all located in the interval $0<u<1$. These zeros will be enumerated in ascending order, with

$$0<u_n(1)<u_n(2)< \cdots <u_n(n-1)<1 \tag{6.1}$$

We shall investigate the asymptotic properties of $u_n(v)$ by first applying the transformation $\vartheta_0 = i\theta_0$ $(0 < \theta_0 < \pi)$ to (4.34). This procedure yields

$$L_n(u) \sim (1/n)[(\theta_0/2)\tan(\theta_0/2)]^{1/2}\left[1 + \sum_{s=1}^{\infty} G_{2s}^{(0)}(\theta_0)(8n)^{-2s}\right]$$

$$\times J_1\left[n\theta_0 + n\theta_0 \sum_{s=1}^{\infty} H_{2s}^{(0)}(\theta_0)(8n)^{-2s}\right] \tag{6.2}$$

as $n \to \infty$, where

$$H_2^{(0)}(\theta_0) = -8\theta_0^{-1}E_1^{(0)}(\theta_0) \tag{6.3}$$

$$G_2^{(0)}(\theta_0) = E_2^{(0)}(\theta_0) + \tfrac{1}{2}[E_1^{(0)}(\theta_0)]^2 + 8\theta_0^{-1}E_1^{(0)}(\theta_0) \tag{6.4}$$

$$H_4^{(0)}(\theta_0) = -8\theta_0^{-1}E_3^{(0)}(\theta_0) + 8\theta_0^{-1}E_1^{(0)}(\theta_0)\,E_2^{(0)}(\theta_0)$$

$$+\tfrac{8}{3}\theta_0^{-1}[E_1^{(0)}(\theta_0)]^3 + 96\theta_0^{-2}[E_1^{(0)}(\theta_0)]^2 \tag{6.5}$$

and the coefficients $E_m^{(0)}(\theta_0)$ $(m = 1, 2, 3)$ are defined in Section 5. The parameter θ_0 is determined by the relation (5.2) with $0 < u < 1$.

We see from (6.2) that $L_n(u)$ will be asymptotically equal to zero when

$$j_{1,v} = n\theta_{0,v}\left[1 + \sum_{s=1}^{\infty} H_{2s}^{(0)}(\theta_{0,v})(8n)^{-2s}\right] \tag{6.6}$$

where $v = 1, 2, \dots, n-1$ and $j_{1,v}$ is the vth zero of the Bessel function $J_1(z)$. If the implicit transcendental Equation (6.6) is solved for the quantity $\theta_{0,v}$, then the vth zero of $L_n(u)$ is given by

$$u_n(v) \sim \tfrac{1}{2}(1 - \cos\theta_{0,v}) \tag{6.7}$$

where $v = 1, 2, \dots, n-1$. This procedure has been carried out for $n = 20$ by applying a direct iterative method to Eq. (6.6) with the coefficients $H_2^{(0)}$ and $H_4^{(0)}$ and an initial solution $\theta_{0,v} \simeq j_{1,v}/n$. The resulting asymptotic values for $u_{20}(v)$ $(v = 1, 2, \dots, 19)$ are compared with the corresponding exact values in Table II. We see that (6.6) gives a highly accurate representation of the zeros $u_{20}(v)$ $(v = 1, 2, \dots, 19)$, especially for small values of v. The turning point at $u = 1$ is responsible for the steady increase in the error $\varepsilon_0(v)$ as v increases.

It is also possible to analyze the asymptotic properties of $u_n(v)$ by applying the transformation $\vartheta_1 = i\theta_1 (0 < \theta_1 < \pi)$ to (4.38). In this manner we find that

$$L_n(u) \sim (-1)^{n-1} n^{-1}[(\theta_1/2)\cot(\theta_1/2)]^{1/2}\left[1 + \sum_{s=1}^{\infty} G_{2s}^{(1)}(\theta_1)(8n)^{-2s}\right]$$

$$\times J_0\left[n\theta_1 + n\theta_1 \sum_{s=1}^{\infty} H_{2s}^{(1)}(\theta_1)(8n)^{-2s}\right] \tag{6.8}$$

Table II. Comparison of the Exact Values of the Zeros $u_{20}(\nu)$ ($\nu = 1, 2, \ldots, 19$) with the Corresponding Asymptotic Values[a]

ν	Exact $u_{20}(\nu)$	$\varepsilon_0(\nu)$	$\varepsilon_1(\nu)$
1	0.009 148 194 729 044	1.1×10^{-14}	1.6×10^{-6}
2	0.030 447 362 919 779	1.3×10^{-13}	1.7×10^{-7}
3	0.063 304 151 925 635	5.8×10^{-13}	3.9×10^{-8}
4	0.106 906 865 018 155	1.8×10^{-12}	1.3×10^{-8}
5	0.160 181 384 291 289	4.6×10^{-12}	5.5×10^{-9}
6	0.221 815 777 023 238	1.0×10^{-11}	2.6×10^{-9}
7	0.290 292 348 346 098	2.0×10^{-11}	1.4×10^{-9}
8	0.363 924 955 120 729	3.9×10^{-11}	7.8×10^{-10}
9	0.440 900 507 350 968	7.3×10^{-11}	4.6×10^{-10}
10	0.519 323 606 421 448	1.3×10^{-10}	2.8×10^{-10}
11	0.597 263 213 834 923	2.5×10^{-10}	1.8×10^{-10}
12	0.672 800 199 236 188	5.0×10^{-10}	1.1×10^{-10}
13	0.744 074 596 517 897	9.1×10^{-10}	7.0×10^{-11}
14	0.809 331 405 095 237	1.9×10^{-9}	4.4×10^{-11}
15	0.866 963 811 441 901	4.3×10^{-9}	2.7×10^{-11}
16	0.915 552 777 215 790	1.2×10^{-8}	1.5×10^{-11}
17	0.953 902 066 951 569	4.0×10^{-8}	7.5×10^{-12}
18	0.981 068 162 968 412	2.2×10^{-7}	2.9×10^{-12}
19	0.996 388 357 181 443	3.9×10^{-6}	5.4×10^{-13}

[a] The quantities $\varepsilon_0(\nu)$ and $\varepsilon_1(\nu)$ are the differences between the asymptotic values of $u_{20}(\nu)$ as determined from the formulas (6.6) and (6.13), respectively, and the exact value of $u_{20}(\nu)$.

as $n \to \infty$, where

$$H_2^{(1)}(\theta_1) = -8\theta_1^{-1} E_1^{(1)}(\theta_1) \tag{6.9}$$

$$G_2^{(1)}(\theta_1) = E_2^{(1)}(\theta_1) + \tfrac{1}{2}[E_1^{(1)}(\theta_1)]^2 \tag{6.10}$$

$$\begin{aligned} H_4^{(1)}(\theta_1) = {} & -8\theta_1^{-1} E_3^{(1)}(\theta_1) + 8\theta_1^{-1} E_1^{(1)}(\theta_1)\, E_2^{(1)}(\theta_1) \\ & + \tfrac{8}{3}\theta_1^{-1}[E_1^{(1)}(\theta_1)]^3 + 32\theta_1^{-2}[E_1^{(1)}(\theta_1)]^2 \end{aligned} \tag{6.11}$$

and the coefficients $E_m^{(1)}(\theta_1)$ ($m = 1, 2, 3$) are defined in Section 5. The parameter θ_1 is determined by the relation (5.12) with $0 < u < 1$. It follows from (6.8) that the zeros of $L_n(u)$ have the alternative asymptotic representation

$$u_n(n - \nu) \sim \tfrac{1}{2}(1 + \cos \theta_{1,\nu}) \tag{6.12}$$

where $\nu = 1, 2, \ldots, n - 1$, $\theta_{1,\nu}$ satisfies the implicit transcendental equation

$$j_{0,\nu} = n\theta_{1,\nu} \left[1 + \sum_{s=1}^{\infty} H_{2s}^{(1)}(\theta_{1,\nu})(8n)^{-2s} \right] \tag{6.13}$$

and $j_{0,\nu}$ is the νth zero of the Bessel function $J_0(z)$. Equation (6.13) has been solved numerically for $n=20$ with the coefficients $H_2^{(1)}$ and $H_4^{(1)}$, and the resulting asymptotic values for $u_{20}(20-\nu)$ ($\nu=1, 2,..., 19$) are compared with the corresponding exact values in Table II. It is clear that (6.13) yields very accurate approximations for the zeros which are close to the turning point $u=1$.

When $n \to \infty$ with ν *fixed* the solution $\theta_{0,\nu}$ of (6.6) will tend to zero as $j_{1,\nu}/n$. For this case we can use the Taylor series

$$H_2^{(0)}(\theta) = \tfrac{2}{15}\theta^2 + \tfrac{1}{63}\theta^4 + O(\theta^6) \tag{6.14}$$

$$H_4^{(0)}(\theta) = -\tfrac{256}{63}\theta^2 - \tfrac{26}{25}\theta^4 + O(\theta^6) \tag{6.15}$$

to derive an asymptotic expansion for $\theta_{0,\nu}$ in powers of $1/n$. The final result is

$$\theta_{0,\nu} = (j_{1,\nu}/n)[1 - (j_{1,\nu}^2/480)\, n^{-4} + (j_{1,\nu}^2/4032)(4 - j_{1,\nu}^2)\, n^{-6} + \cdots] \tag{6.16}$$

If we substitute (6.16) in (6.7), we find that

$$\begin{aligned} u_n(\nu) \sim \tfrac{1}{4}(j_{1,\nu}/n)^2\, &[1 - (j_{1,\nu}^2/12)\, n^{-2} \\ &+ (j_{1,\nu}^2/720)(-3 + 2j_{1,\nu}^2)\, n^{-4} \\ &+ (j_{1,\nu}^2/20160)(40 + 4j_{1,\nu}^2 - j_{1,\nu}^4)\, n^{-6} + \cdots] \end{aligned} \tag{6.17}$$

as $n \to \infty$ with ν *fixed.* This expansion is consistent with the numerical work of Majumdar[30] and the scaling theory arguments of Gaunt[28] for *general* Ising model systems, provided that we take $u_c \equiv 0$ and the critical exponent $\Delta = \frac{1}{2}$.

In a similar manner we can use the Taylor series

$$H_2^{(1)}(\theta) = -\tfrac{16}{3} - \tfrac{22}{45}\theta^2 + O(\theta^4) \tag{6.18}$$

$$H_4^{(1)}(\theta) = \tfrac{2176}{45} + \tfrac{40736}{2835}\theta^2 + O(\theta^4) \tag{6.19}$$

to obtain the following expansion for the solution $\theta_{1,\nu}$ of the transcendental equation (6.13):

$$\theta_{1,\nu} = (j_{0,\nu}/n)[1 + (1/12)\, n^{-2} + (1/1440)(11 j_{0,\nu}^2 - 7)\, n^{-4} + \cdots] \tag{6.20}$$

The substitution of (6.20) in (6.12) yields the further asymptotic representation

$$u_n(n-v) \sim 1 - (j_{0,v}^2/4)\, n^{-2} + (j_{0,v}^2/48)(-2 + j_{0,v}^2)\, n^{-4} + (j_{0,v}^2/2880)(2 + 9j_{0,v}^2 - 2j_{0,v}^4)\, n^{-6} + \cdots \tag{6.21}$$

as $n \to \infty$ with v *fixed.*

Finally, we define $M_n(a, b)$ to be the number of zeros $u_n(v)$ $(v = 1, 2,..., n-1)$ which lie in the interval (a, b), where $0 \leqslant a < b \leqslant 1$. From the asymptotic theory of orthogonal polynomials[31] it can be shown that

$$M_n(a, b) \sim \int_a^b \rho(u)\, du \tag{6.22}$$

as $n \to \infty$, where

$$\rho(u) = (n-1)\, \pi^{-1}[u(1-u)]^{-1/2} \tag{6.23}$$

7. ASYMPTOTIC BEHAVIOR OF $L_n(u)$ AS $u \to 0+$ AND $u \to 1-$

The behavior of $L_n(u)$ as $n \to \infty$ and $u \to 0+$ may be determined by applying the Taylor series

$$E_1^{(0)}(\theta) = -\tfrac{1}{60}\theta^3 - \tfrac{1}{504}\theta^5 + O(\theta^7) \tag{7.1}$$

$$E_2^{(0)}(\theta) = -\tfrac{1}{63}\theta^4 - \tfrac{1}{288}\theta^6 + O(\theta^8) \tag{7.2}$$

$$E_3^{(0)}(\theta) = \tfrac{32}{63}\theta^3 + \tfrac{2}{15}\theta^5 + O(\theta^7) \tag{7.3}$$

and (5.2) to the basic result (5.1). In this manner we eventually find that

$$L_n(u)/u \sim \sum_{m=0}^{\infty} \psi_m^{(0)}(\xi_0)[(2m)!]^{-1}\, n^{-2m} \tag{7.4}$$

as $n \to \infty$ and $u \to 0+$, where

$$\psi_0^{(0)}(\xi_0) = (2/\xi_0)\, J_1(\xi_0) \tag{7.5}$$

$$\psi_1^{(0)}(\xi_0) = (\xi_0/6)[3J_1(\xi_0) - \xi_0 J_2(\xi_0)] \tag{7.6}$$

$$\psi_2^{(0)}(\xi_0) = (\xi_0^2/120)[\xi_0(123 - 5\xi_0^2)\, J_1(\xi_0) - (12 + 42\xi_0^2)\, J_2(\xi_0)] \tag{7.7}$$

and

$$\xi_0 = 2nu^{1/2} \tag{7.8}$$

It is also possible to derive a *general* formula for the scaling functions $\psi_m^{(0)}(\xi_0)$ $(m=0, 1, 2,...)$ by first using the hypergeometric representation (2.10) to write $L_n(u)$ in the form

$$L_n(u)=\frac{u}{n}\sum_{k=0}^{n-1}\frac{\Gamma(n+k+1)}{(2)_k\,\Gamma(n-k)}\frac{(-u)^k}{k!} \tag{7.9}$$

If the asymptotic expansion[25]

$$\frac{\Gamma(n+k+1)}{\Gamma(n-k)}\sim n^{2k+1}\sum_{m=0}^{\infty}\frac{(-2k-1)_{2m}\,B_{2m}^{(2k+2)}(k+1)}{(2m)!\,n^{2m}} \tag{7.10}$$

as $n\to\infty$ is now substituted in (7.9), we obtain the closed-form expression

$$\psi_m^{(0)}(\xi_0)=\sum_{k=0}^{\infty}\frac{(-2k-1)_{2m}\,B_{2m}^{(2k+2)}(k+1)}{(k+1)!}\frac{(-\xi_0^2/4)^k}{k!} \tag{7.11}$$

where $B_k^{(a)}(x)$ denotes a generalized Bernoulli polynomial. It can be shown that (7.11) is consistent with the results (7.5)–(7.7).

In a similar manner the application of the Taylor series

$$E_1^{(1)}(\theta)=\tfrac{2}{3}\theta+\tfrac{11}{180}\theta^3+O(\theta^5) \tag{7.12}$$

$$E_2^{(1)}(\theta)=\tfrac{4}{15}\theta^2+\tfrac{37}{630}\theta^4+O(\theta^6) \tag{7.13}$$

$$E_3^{(1)}(\theta)=-\tfrac{64}{15}\theta-\tfrac{376}{315}\theta^3+O(\theta^5) \tag{7.14}$$

and (5.12) to (5.11) yields the asymptotic expansion

$$L_n(u)/u\sim(-1)^{n-1}n^{-1}\sum_{m=0}^{\infty}\psi_m^{(1)}(\xi_1)[(2m)!]^{-1}n^{-2m} \tag{7.15}$$

as $n\to\infty$ and $u\to 1-$, where

$$\psi_0^{(1)}(\xi_1)=J_0(\xi_1) \tag{7.16}$$

$$\psi_1^{(1)}(\xi_1)=(\xi_1/12)[5\xi_1 J_0(\xi_1)+(2-\xi_1^2)\,J_1(\xi_1)] \tag{7.17}$$

$$\psi_2^{(1)}(\xi_1)=(\xi_1/240)[(24\xi_1+291\xi_1^3-5\xi_1^5)\,J_0(\xi_1)+(-48+144\xi_1^2-72\xi_1^4)\,J_1(\xi_1)] \tag{7.18}$$

and

$$\xi_1=2n(1-u)^{1/2} \tag{7.19}$$

A general formula for $\psi_m^{(1)}(\xi_1)$ $(m=0, 1, 2,...)$ can be derived by applying the relation[27]

$$P_n^{(1,0)}(x)=(-1)^n\,{}_2F_1(-n, n+2; 1; \tfrac{1}{2}+\tfrac{1}{2}x) \tag{7.20}$$

to (2.8). In this manner we obtain

$$L_n(u) = \frac{(-1)^n u}{n^2} \sum_{k=0}^{n-1} \frac{\Gamma(n+k+1)}{(1)_k \Gamma(n-k)} \frac{[-(1-u)]^k}{k!} \tag{7.21}$$

The substitution of the asymptotic expansion (7.10) in this result gives the required closed-form expression

$$\psi_m^{(1)}(\xi_1) = \sum_{k=0}^{\infty} (-2k-1)_{2m} B_{2m}^{(2k+2)}(k+1)(-\xi_1^2/4)^k/(k!)^2 \tag{7.22}$$

where $B_k^{(a)}(x)$ denotes a generalized Bernoulli polynomial. It has been verified that (7.22) is consistent with the results (7.16)–(7.18).

8. CONCLUDING REMARKS

In this paper we have established uniform asymptotic representations for the high-field polynomials $L_n(u)$ of the one-dimensional spin $\frac{1}{2}$ Ising model. These representations have been used to derive the following asymptotic expansions for the zeros $u_n(v)$ $(v = 1, 2,..., n-1)$:

$$u_n(v) \sim A_0(v)\, n^{-2} \left[1 - \sum_{m=1}^{\infty} B_m(v)\, n^{-2m} \right] \tag{8.1}$$

$$u_n(n-v) \sim 1 - \sum_{m=1}^{\infty} C_m(v)\, n^{-2m} \tag{8.2}$$

as $n \to \infty$, with v *fixed*, where the coefficients $B_m(v)$ and $C_m(v)$ $(m = 1, 2,...)$ are polynomials in the Bessel function zeros $j_{1,v}$ and $j_{0,v}$, respectively.

For spin $\frac{1}{2}$, nearest neighbor Ising models on a d-dimensional lattice Ω_d with $d > 1$ it appears from the numerical work of Majumdar[30] and Gaunt[28] that $L_n(u)$ still has exactly $n-1$ real zeros $u_n(v)$ $(v = 1, 2,..., n-1)$ which lie in the interval $u_c < u < 1$. However, for $d > 1$ there are additional zeros of $L_n(u)$ which lie either on the negative real u axis or in the complex u plane with $\mathrm{Im}(u) \neq 0$. (There is also a trivial multiple zero at $u = 0$.) In these higher-dimensional models one can use critical point scaling theory to obtain the leading-order asymptotic formula[28]

$$u_n(v) - u_c \sim A_0(v, \Omega_d)\, n^{-1/\Delta} \tag{8.3}$$

as $n \to \infty$ with v *fixed*, where the amplitude $A_0(v, \Omega_d)$ depends on the lattice Ω_d and Δ is a standard critical exponent which depends only on the dimensionality d of the lattice. It is clear that the formula (8.3) agrees to leading order with (8.1) provided that we take $u_c \equiv 0$ and $\Delta = \frac{1}{2}$.

Recently, D. S. Gaunt (private communication) has also shown that the asymptotic formula

$$u_n(n-v) \sim 1 - C_1(v, \Omega_d)\, n^{-2} \tag{8.4}$$

as $n \to \infty$ with v *fixed* is valid for *all* spin $\frac{1}{2}$ nearest neighbor Ising models with $d \geqslant 1$. From this result it is reasonable to expect that the *general form* of the expansion (8.2) will be applicable to other higher-dimensional Ising models. I hope to investigate this conjecture in a future publication.

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